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FATOU PROPERTIES OF MONOTONE SEMINORMS ON RIESZ SPACES(1)

BY

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ABSTRACT. A monotone seminorm ρ on a Riesz space L is called σ-Fatou if $\rho(u_n)\uparrow\rho(u)$ holds for every $u \in L^+$ and sequence $\{u_n\}$ in L satisfying $0 \leq u_n\uparrow u$. A monotone seminorm ρ on L is called strong Fatou if $\rho(u_\nu)\uparrow\rho(u)$ holds for every $u \in L^+$ and directed system $\{u_\nu\}$ in L satisfying $0 \leq u_\nu\uparrow u$. In this paper we determine those Riesz spaces L which have the property that, for any monotone seminorm ρ on L, the largest strong Fatou seminorm ρ_m majorized by ρ is of the form: $\rho_m(f) = \inf\{\sup_\nu \rho(u_\nu)\colon 0 \leq u_\nu \uparrow |f|\}$ for $f \in L$. We discuss, in a Riesz space L, the condition that a monotone seminorm ρ as well as its Lorentz seminorm ρ_L is σ-Fatou in terms of the order and relative uniform topologies on L. A parallel discussion is also given for outer measures on Boolean algebras.

1. **Notation.** It is convenient to first introduce some notation. Indices from a countable set will be denoted by $m, n, k, \cdots$, from an arbitrary set by $\kappa, \nu, \mu, \cdots$. Let X be any partially ordered set. If $x \in X$, and if a sequence $\{x_n\}$ in X satisfies $x_n\uparrow x$, and if for each n a sequence $\{x_{nk}\colon k = 1, 2, \cdots\}$ satisfies $x_{nk}\uparrow_k x_n$, then we will write $x_{nk}\uparrow x_n\uparrow x$. If $x \in X$, and if an upwards directed system $\{x_\nu\colon \nu \in V\}$ satisfies $x_\nu\uparrow x$, and if for each $\nu \in V$ an index set K_ν and an upwards directed system $\{x_{\kappa_\nu}\colon \kappa_\nu \in K_\nu\}$ satisfy $x_{\kappa_\nu}\uparrow_{\kappa_\nu \in K_\nu} x$, then we will write $x_{\kappa_\nu}\uparrow x_\nu\uparrow x$.

2. **Outer measures on Boolean algebras.** A real function ϕ on a Boolean algebra B is called a finitely additive measure if (i) $0 \leq \phi(a) < \infty$ for all $a \in B$, (ii) $\phi(a \vee b) = \phi(a) + \phi(b)$ whenever $a, b \in B$ and a, b are disjoint, (iii) $\phi(1) \neq 0$. A finitely additive measure ϕ on B is called countably additive if $\phi(\bigvee_1^\infty a_n) = \Sigma_1^\infty \phi(a_n)$ for every mutually disjoint countable subset $\{a_1, a_2, \cdots\}$ of B such that $\bigvee_1^\infty a_n$ exists. A finitely additive measure ϕ on B is called purely finitely additive if every countably additive measure ϕ' such that $\phi' \leq \phi$ is identically zero. In [11], K. Yosida and E. Hewitt proved that every

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finitely additive measure ϕ on a Boolean algebra can be uniquely written as the sum of a countably additive measure ϕ_c and a purely finitely additive measure ϕ_p. For every finitely additive measure ϕ on a Boolean algebra B, a related function ϕ_L on B is defined:

$$\phi_L(a) = \inf \left\{\lim_n \phi(a_n): a_n \uparrow a\right\} \quad \text{for all } a \in B.$$

It follows that ϕ_L is a finitely additive measure on B, and ϕ is countably additive if and only if $\phi_L = \phi$. A result due to M. A. Woodbury [10] and H. Bauer [1] states that, for every finitely additive measure ϕ on a Boolean algebra B, its countably additive part ϕ_c is equal to ϕ_L. Note that $\phi_c = \phi_L$ is equivalent to the statement that $\phi_L = \phi_{LL}$.

If ϕ is a finitely additive measure on a Boolean algebra B, then ϕ is monotone, i.e., $a \leqslant b$ implies $\phi(a) \leqslant \phi(b)$. A function $\rho: B \longrightarrow [0, \infty]$ is called an outer measure on B if (i) ρ is monotone, (ii) $\rho(0) = 0$, (iii) $a \leqslant \bigvee_1^\infty a_n$ implies $\rho(a) \leqslant \Sigma_1^\infty \rho(a_n)$. Given any function $\rho: B \longrightarrow [0, \infty]$ which is monotone, ρ_L is defined as:

$$\rho_L(a) = \inf \left\{\lim_n \rho(a_n): a_n \uparrow a\right\} \quad \text{for } a \in B.$$

If we consider the class of all outer measures on a Boolean algebra instead of the class of all finitely additive measures in the above result of Woodbury and Bauer, then the situation is somewhat different. J. A. R. Holbrook proved that if B is a Boolean algebra then, for every outer measure ρ on B, $\rho_L = \rho_{LL}$ if and only if B has the Egoroff property [3].

In the following, we shall replace sequences by directed systems and obtain a result similar to that of Holbrook. Note that an element a of a Boolean algebra B has the Egoroff property if and only if, whenever $a_{nk} \uparrow a_n \uparrow a$, there exists a sequence $b_m \uparrow a$ such that, for every m, $b_m \leqslant a_{n(m)k(m)}$ for some appropriate index $n(m), k(m)$ (the proof of this is similar to the proof of Lemma 2.2 in [2]).

DEFINITION 2.1. An element a of a Boolean algebra B is said to have the generalized Egoroff property, whenever $a_{\kappa_\nu} \uparrow a_\nu \uparrow a$, there exists an upwards directed system $\{b_\mu\}$ such that $b_\mu \uparrow a$ and, for every μ, there exists a $\nu = \nu(\mu)$ and a κ_ν in $\{\kappa_{\nu(\mu)}\}$ such that $b_\mu \leqslant a_{\kappa_\nu}$. A Boolean algebra is said to have the generalized Egoroff property if every one of its elements has the generalized Egoroff property.

For every monotone function $\rho: B \longrightarrow [0, \infty]$, the function ρ_l is defined by:

$$\rho_l(a) = \inf \left\{\sup_\nu \rho(a_\nu): a_\nu \uparrow a\right\} \quad \text{for } a \in B.$$

It follows that ρ_l is a monotone function on B, and $\rho_l = \rho$ if and only if for every $a \in B$ $a_\nu \uparrow a$ implies $\sup_\nu \rho(a_\nu) = \rho(a)$.

THEOREM 2.2. *Let B be a Boolean algebra. Then $\rho_l = \rho_{ll}$ for every outer measure ρ on B if and only if B has the generalized Egoroff property. In fact,*

(1) *if B has the generalized Egoroff property, then $\rho_l = \rho_{ll}$ for every monotone function $\rho: B \longrightarrow [0, \infty]$;*

(2) *if $\rho_l = \rho_{ll}$ for every finite-valued outer measure ρ on B, then B has the generalized Egoroff property.*

PROOF. (1) Assume that B has the generalized Egoroff property. Let ρ be a monotone function on B. It is clear that $\rho_l \geqslant \rho_{ll}$. On the other hand, suppose $\rho_{ll}(a) < \alpha$; in this case, there must exist a directed system $a_\nu \uparrow a$ and for each ν a directed system $a_{\kappa_\nu} \uparrow_{\kappa_\nu} a_\nu$ such that $\rho(a_{\kappa_\nu}) < \alpha$ for all κ_ν. Since $a_{\kappa_\nu} \uparrow a_\nu \uparrow a$ and B has the generalized Egoroff property, there exists $b_\mu \uparrow a$ such that, for every μ, $b_\mu \leqslant a_{\kappa_\nu}$ for some κ_ν. Thus $\rho(b_\mu) \leqslant \rho(a_{\kappa_\nu}) < \alpha$ for all μ, so that $\rho_l(a) \leqslant \alpha$. Since this holds for any α satisfying $\rho_{ll}(a) < \alpha$, we therefore have $\rho_l(a) \leqslant \rho_{ll}(a)$.

(2) It is sufficient to show that the unit element 1 of B has the generalized Egoroff property. Let $a_{\kappa_\nu} \uparrow a_\nu \uparrow 1$. We may assume that $a_\nu \neq 1$ for all ν. Define a function ρ on B as follows: for each $c \in B$,

$$\begin{aligned} \rho(c) &= 0 \quad \text{if } c = 0, \\ &= \tfrac{1}{2} \quad \text{if } 0 < c \leqslant a_{\kappa_\nu} \text{ for some } \kappa_\nu, \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

It is evident that ρ is a finite-valued outer measure on B. Moreover, since $a_{\kappa_\nu} \uparrow a_\nu \uparrow 1$ and $\rho(a_{\kappa_\nu}) = \tfrac{1}{2}$ for all κ_ν, we have $\rho_{ll}(1) = \tfrac{1}{2}$. Then $\rho_l(1) = \tfrac{1}{2}$ by the assumption. This means that there exists a directed system $b_\mu \uparrow 1$ such that for every μ, $\rho(b_\mu) = \tfrac{1}{2}$, thus, for every μ, there is some κ_ν such that $b_\mu \leqslant a_{\kappa_\nu}$. Therefore, the unit element has the generalized Egoroff property and the proof of the theorem is complete.

Let B be a Boolean algebra. For every pair of elements a, b of B, the symmetric difference $(a' \wedge b) \vee (a \wedge b')$ of a and b will be denoted by $a \vartriangle b$. We say that a sequence $\{a_n\}$ in B is order convergent to an element $a \in B$ if there exists a sequence $b_n \downarrow 0$ such that $a_n \vartriangle a \leqslant b_n$ for all n; this will be denoted by $a_n \longrightarrow a$. A subset A of B is said to be order closed if, for every order convergent sequence in A, the order limit of the sequence is also a member of A. The topology on B which has as closed sets the family of all order closed sets is called the order topology of B. For any subset A of B, the set of all $a \in B$ such that there exists a sequence in A converging in order to a is called the pseudo order closure of A and it will be denoted by A'. In [2] it is proved that a Boolean algebra B has the Egoroff property if and only if A' is order closed for every subset A of B. From this, together with Holbrook's result, we have

THEOREM 2.3. *If B is a Boolean algebra then $\rho_L = \rho_{LL}$ for every outer measure ρ on B if and only if A' is order closed for every subset A of B.*

We shall next look at a single outer measure ρ on a Boolean algebra B and obtain a necessary and sufficient condition that $\rho_L = \rho_{LL}$ in terms of the order topology of B. For every outer measure ρ on B and real $\alpha > 0$, we shall denote by $A(\rho, \alpha)$ the set of all elements $a \in B$ such that $\rho(a) \leqslant \alpha$.

LEMMA 2.4. *Let ρ be an outer measure on a Boolean algebra B. Then $\rho = \rho_L$ if and only if $A(\rho, \alpha)$ is order closed for every real $\alpha > 0$.*

PROOF. Assume that $\rho = \rho_L$. Let $\alpha > 0$. If $a_n \in A(\rho, \alpha)$ for $n = 1, 2, \cdots$ and $a_n \longrightarrow a$, then there exists $b_n \downarrow 0$ such that $a_n \Delta a \leqslant b_n$ for all n. The sequence $c_n = a \wedge b_n'$ satisfies $c_n \uparrow a$ and $c_n \leqslant a_n$ for all n. By the assumption that $\rho = \rho_L$, we have $\rho(c_n) \uparrow p(a)$ and so $a \in A(\rho, \alpha)$. This proves that $A(\rho, \alpha)$ is order closed. Conversely, assume that for every $\alpha > 0, A(\rho, \alpha)$ is order closed. Let $a_n \uparrow a$. If a real number α is such that $\rho(a_n) < \alpha$ for all n, then $a_n \in A(\rho, \alpha)$ for all n and so, by the assumption that $A(\rho, \alpha)$ is order closed, $a \in A(\rho, \alpha)$, i.e., $\rho(a) \leqslant \alpha$. This proves that $\rho = \rho_L$.

LEMMA 2.5. *Let ρ be an outer measure on a Boolean algebra B. Then $A(\rho_L, \alpha) = \bigcap_{\epsilon > 0} A'(\rho, \alpha + \epsilon)$.*

PROOF. Let $a \in A(\rho_L, \alpha)$. Then, for every $\epsilon > 0, \rho_L(a) < \alpha + \epsilon$ and so there exists $a_n \uparrow a$ such that $\rho(a_n) < \alpha + \epsilon$ for all n. Hence $a \in \bigcap_{\epsilon > 0} A'(\rho, \alpha + \epsilon)$. Conversely, let $a \in \bigcap_{\epsilon > 0} A'(\rho, \alpha + \epsilon)$. Then for each $\epsilon > 0, a \in A'(\rho, \alpha + \epsilon)$ and so there exists $a_n \uparrow a$ such that $a_n \in A(\rho, \alpha + \epsilon)$ for all n. Thus $\rho_L(a) \leqslant \alpha + \epsilon$ for all $\epsilon > 0$ and hence $\rho_L(a) \leqslant \alpha$, i.e., $a \in A(\rho_L, \alpha)$.

The following result is a direct consequence of Lemmas 2.4 and 2.5.

THEOREM 2.6. *Let ρ be an outer measure on a Boolean algebra B. Then $\rho_L = \rho_{LL}$ if and only if the set $\bigcap_{\epsilon > 0} A'(\rho, \alpha + \epsilon)$ is order closed for every $\alpha > 0$.*

3. Strong Fatou property of monotone seminorms on Riesz spaces. An extended real valued function ρ on a Riesz space L is called a monotone seminorm on L if, for every $f, g \in L$,

(i) $0 \leqslant \rho(f) \leqslant \infty, \rho(f + g) \leqslant \rho(f) + \rho(g)$, and $\rho(\lambda f) = \lambda \rho(f)$ for all real $\lambda \geqslant 0$,

(ii) ρ is monotone, i.e., $|f| \leqslant |g|$ implies $\rho(f) \leqslant \rho(g)$. It follows from (ii) that $\rho(f) = \rho(|f|)$ for all $f \in L$. A monotone seminorm ρ on L is called σ-Fatou if $0 \leqslant u_n \uparrow u$ implies that $\rho(u_n) \uparrow \rho(u)$. A given monotone seminorm ρ may not itself be σ-Fatou, but, among those σ-Fatou monotone seminorms majorized by ρ, the largest element ρ_M always exists. In fact,

$$\rho_M(f) = \sup \{\rho'(f)\colon \ \rho' \text{ is } \sigma\text{-Fatou monotone seminorm such that } \rho' \leqslant \rho\}.$$

It is not known, in general, how to construct ρ_M explicitly in terms of ρ. However, there are three cases of interest in which ρ_M may be constructed explicitly. To facilitate the discussion, we define the Lorentz seminorm ρ_L associated with a given monotone seminorm ρ. If ρ is a monotone seminorm on L, then ρ_L is defined by: for every $f \in L$,

$$\rho_L(f) = \inf \left\{ \lim_n \rho(u_n)\colon \ 0 \leqslant u_n \uparrow |f| \right\}.$$

We can see easily that, for any given monotone seminorm ρ on L, ρ_L is again a monotone seminorm on L; moreover, $\rho \geqslant \rho_L \geqslant \rho_M$, $\rho_L = \rho$ if and only if ρ is σ-Fatou, $\rho_M = \rho_L$ if and only if ρ_L is σ-Fatou. Theorem 7.3 of [6] states that, if the Riesz space L is a real Banach space, then $\rho_M = \rho_L$ for every monotone seminorm ρ on L. Theorem 20.4 of [6] states that, if a monotone seminorm ρ on a Riesz space L is of the form $\rho(f) = \phi(|f|)$ for every $f \in L$, where ϕ is a positive linear functional on L, then $\rho_M = \rho_L$. The third result is due to J. A. R. Holbrook. In [4], he proved that if L is a Riesz space then $\rho_M = \rho_L$ for every monotone seminorm ρ on L if and only if L has the almost Egoroff property.

DEFINITION 3.1. An element f of a Riesz space L is said to have the almost Egoroff property if, given any real number ϵ with $0 < \epsilon < 1$ and any countable set of sequences $\{u_{nk}\colon k = 1, 2, \cdots\}$, $n = 1, 2, \cdots$, in L such that $0 \leqslant u_{nk} \uparrow_k |f|$ for all n, there exists a sequence $0 \leqslant v_m^\epsilon \uparrow_m |f|$ and for every m, n of indices an index $k(m, n)$ such that $(1-\epsilon)v_m^\epsilon \leqslant u_{n\,k(m,n)}$. A Riesz space is said to have the almost Egoroff property if every one of its elements has the almost Egoroff property.

We shall derive a similar result to that of Holbrook, with sequences replaced by direct sets. In order to do this, we first give a modified proof of Holbrook's result.

For every subset S of a Riesz space L, we shall denote by $\langle S\rangle$ the convex hull of S.

LEMMA 3.2. *Let L be a Riesz space and $u \in L^+$. Then the following two statements are equivalent.*

(1) *u has the almost Egoroff property.*

(2) *If $0 \leqslant u_{nk} \uparrow_k u_n \uparrow u$ and $0 < \epsilon < 1$, then there exists a sequence $\{v_m^\epsilon\}$ with $0 \leqslant v_m^\epsilon \uparrow_m u$ and such that, for every m, $(1-\epsilon)v_m^\epsilon \leqslant z_m$ for some elements z_m in $\langle\{u_{nk}\}\rangle$.*

PROOF. (1)$\Rightarrow$(2): Let $0 \leqslant u_{nk} \uparrow_k u_n \uparrow u$ and $0 < \epsilon < 1$. Then $0 \leqslant u_{nk} + u - u_n \uparrow_k u$ and so by (1) there exists a sequence $\{w_m^\epsilon\}$ with $0 \leqslant w_m^\epsilon \uparrow_m u$ and $(1-\epsilon)w_m^\epsilon \leqslant u_{n,k(m,n)} + u - u_n$ for some $k(m, n)$. Set $v_m^\epsilon = (1-\epsilon)^{-1}[(1-\epsilon)w_m^\epsilon - u + u_m]^+$; then we have $0 \leqslant v_m^\epsilon \uparrow_m u$ and $(1-\epsilon)v_m^\epsilon \leqslant u_{m,k(m,m)}$. Hence (2) holds.

(2)$\Rightarrow$(1): Let $0 \leqslant u_{nk} \uparrow_k u$ and $0 < \epsilon < 1$. We may assume that $u_{nk} \downarrow_n$ as we can always replace u_{nk} by $u_{1k} \wedge \cdots \wedge u_{nk}$. We may also assume that there exists a sequence $\{u_n\}$ in L^+ such that: (i) $0 \leqslant u_n \uparrow u$, (ii) $u_n \not\geqslant (1-\epsilon)u$ for all n. For if such a sequence $\{u_n\}$ does not exist, then $0 \leqslant u_{nk} \uparrow_k u$ implies that for every n there exists some index $k(n)$ such that $u_{n,k(n)} \geqslant (1-\epsilon)u$, and so the sequence $\{v_m^\epsilon\}$ with $v_m^\epsilon = u$ for all m satisfies the required condition.

Now $0 \leqslant u_{nk} \wedge u_n \uparrow_k u_n \uparrow u$. If we set $\delta = \epsilon/2$, then by (2), there exists a sequence $\{w_m^\delta\}$ with $0 \leqslant w_m^\delta \uparrow_m u$ and such that, for every m, $(1-\delta)w_m^\delta \leqslant z_m$ for some element z_m in $\langle\{u_{nk} \wedge u_n\}\rangle$. For a fixed m, $z_m \in \langle\{u_{nk} \wedge u_n\}\rangle$ means that we can write $z_m = \Sigma_{n,k}\alpha_{nk}^m(u_{nk} \wedge u_n)$ where α_{nk}^m $(n, k = 1, 2, \cdots)$ are nonnegative real numbers, zero except for finitely many n and k, and such that $\Sigma_{n,k}\alpha_{nk}^m = 1$. For every fixed pair of m and n, write $h(m, n) = \max\{k: \alpha_{nk}^m \neq 0\}$ and $\alpha_n^m = \Sigma_k \alpha_{nk}^m$; then, for a fixed m, $\alpha_n^m \geqslant 0$ for all n with $\alpha_n^m = 0$ except for finitely many n and $\Sigma_n \alpha_n^m = 1$. Furthermore, we have

$$(1-\delta)w_m^\delta \leqslant z_m \leqslant \sum_n \alpha_n^m(u_{n,h(m,n)} \wedge u_n).$$

We will show that the sequence $v_m^\epsilon = (1-\epsilon)^{-1}[2(1-\delta)w_m^\delta - u]^+$, $m = 1, 2, \cdots$, satisfies $0 \leqslant v_m^\epsilon \uparrow_m u$ and for a given pair, M and N, of indices, there exists an index $k(M, N)$ such that $(1-\epsilon)v_M^\epsilon \leqslant u_{N,k(M,N)}$.

Obviously $0 \leqslant v_m^\epsilon \uparrow_m u$. Given M and N, let $\gamma = \sup\{\Sigma_{n \geqslant N}\alpha_n^m : m \geqslant M\}$. We prove that $\gamma > ½$. It is clear that $0 \leqslant \gamma \leqslant 1$. Moreover, for $m \geqslant M$, we have

$$u - (1-\delta)w_m^\delta \geqslant \sum_n \alpha_n^m(u - u_{n,h(m,n)} \wedge u_n) \geqslant \sum_{n<N} \alpha_n^m(u - u_n)$$

$$\geqslant \left(\sum_n \alpha_n^m - \sum_{n<N} \alpha_n^m\right)(u - u_N) \geqslant (1-\gamma)(u - u_N).$$

Since $\inf\{u - (1-\delta)w_m^\delta : m \geqslant M\} = \delta u$, so $\delta u \geqslant (1-\gamma)(u - u_N)$ and we have either $\gamma = 1$ or $u_N \geqslant (1 - \delta/(1-\gamma))u$. Recall that $u_n \not\geqslant (1-\epsilon)u$ for all n; hence $u_N \geqslant (1 - \delta/(1-\gamma))u$ implies that $\delta/(1-\gamma) > \epsilon$, i.e., $\gamma > ½$. From the above argument, there exists an index $p \geqslant M$ such that $\Sigma_{n \geqslant N}\alpha_n^p \geqslant ½$ and so

$$u - (1-\delta)w_M^\delta \geqslant u - (1-\delta)w_p^\delta \geqslant \sum_n \alpha_n^p(u - u_{n,h(p,n)} \wedge u_n)$$
$$\geqslant \sum_{n \geqslant N} \alpha_n^p(u - u_{n,h(p,n)});$$

if we let $k(M, N) = \max\{h(p, n)\colon \alpha_n^p \neq 0\}$ and recall that $u_{nk}\downarrow_n$, we have

$$u - (1-\delta)w_M^\delta \geqslant \left(\sum_{n \geqslant N} \alpha_n^p\right)(u - u_{N,k(M,N)}) \geqslant ½(u - u_{N,k(M,N)}),$$

and hence $2(1-\delta)w_M^\delta - u \leqslant u_{N,k(M,N)}$; thus $(1-\epsilon)^\epsilon_M \leqslant u_{N,k(M,N)}$ as required.

THEOREM 3.3. *Let L be a Riesz space and $u \in L^+$.*

(1) *If u has the almost Egoroff property, then $\rho_M(u) = \rho_L(u)$ for every monotone seminorm ρ on L.*

(2) *If $\rho_M(u) = \rho_L(u)$ for every monotone seminorm ρ on L such that $\rho(u) < \infty$, then u has the almost Egoroff property.*

PROOF. Note that $\rho_M = \rho_L$ if and only if $\rho_L = \rho_{LL}$.

(1) It is clear that $\rho_{LL}(u) \leqslant \rho_L(u)$. On the other hand, suppose $\rho_{LL}(u) < \alpha$; in this case, there must exist a sequence $0 \leqslant u_n \uparrow u$ and, for every n, a sequence $0 \leqslant u_{nk}\uparrow_k u_n$ such that $\rho(u_{nk}) < \alpha$ for all n, k. By the assumption that u has the almost Egoroff property and Lemma 3.2, there exists , for every $0 < \epsilon < 1$, a sequence $0 \leqslant v_m^\epsilon \uparrow u$ such that, for every m, $(1-\epsilon)^\epsilon_m \leqslant z_m$ for some z_m in $\langle\{u_{nk}\}\rangle$. We then have $(1-\epsilon)\rho(v_m^\epsilon) \leqslant \rho(z_m) < \alpha$ for all m; hence $\rho_L(u) \leqslant (1-\epsilon)^{-1}\alpha$. Since this is true for every ϵ with $0 < \epsilon < 1$, we thus obtain $\rho_L(u) \leqslant \alpha$. Therefore $\rho_L(u) \leqslant \rho_{LL}(u)$.

(2) It is sufficient to show that if $0 \leqslant u_{nk}\uparrow u_n \uparrow u$ and $0 < \epsilon < 1$, then there is a sequence $0 \leqslant v_m^\epsilon \uparrow u$ such that, for every m, $(1-\epsilon)v_m^\epsilon \leqslant z_m$ for some z_m in $\langle\{u_{nk}\}\rangle$.

Let $0 \leqslant u_{nk}\uparrow u_n \uparrow u$ and $0 < \epsilon < 1$. Set $\epsilon_1 = \epsilon/2$ and define a function ρ on L as follows: for every f in L,

$$\rho(f) = \inf\Big\{\sum_{n,k} \alpha_{nk}\colon \alpha_{nk} \geqslant 0 \text{ for all } n, k,\ \alpha_{nk} = 0 \text{ except}$$
$$\text{for finitely many } n, k \text{ and } \sum_{n,k} \alpha_{nk}(u_{nk} \vee \epsilon_1 u) \geqslant |f|\Big\},$$
$$= \infty \text{ if there is no such finite sum covering } |f|.$$

It can be easily verified that ρ is a monotone seminorm on L. Moreover, $\rho(u) < \infty$; in fact, $\rho(u) \leqslant \epsilon_1^{-1}$.

Now $\rho(u_{nk}) \leqslant 1$ for all n, k; then $\rho_L(u_n) \leqslant 1$ for all n and so $\rho_{LL}(u) \leqslant 1$. By the assumption that $\rho_L(u) = \rho_{LL}(u)$, we then have $\rho_L(u) \leqslant 1$. Set $\epsilon_2 =$

$(2-\epsilon)^{-1}\epsilon$. Then $\rho_L(u) < 1 + \epsilon_2$ implies that there exists a sequence $0 \leq w_m \uparrow u$ and, for each m, $\rho(w_m) < 1 + \epsilon_2$. If we let $v_m^\epsilon = (1-\epsilon)^{-1}(w_m - \epsilon u)^+$, we have $0 \leq v_m^\epsilon \uparrow u$. It remains to be shown that, for every m, $(1-\epsilon)v_m \leq z_m$ for some element z_m in $\langle\{u_{nk}\}\rangle$.

For each m, since $\rho(w_m) < 1 + \epsilon_2$, there exist, by the definition of ρ, real numbers $\alpha_{nk}^m, n, k = 1, 2, \cdots$, such that $\alpha_{nk}^m \geq 0$ for all n, k, $\alpha_{nk}^m = 0$ except for finitely many n, k, $0 < \Sigma_{n,k}\alpha_{nk}^m < 1 + \epsilon_2$ and $\Sigma_{n,k}\,\alpha_{nk}^m(u_{nk} \vee \epsilon_1 u) \geq w_m$. We then have

$$(1 + \epsilon_2)^{-1}w_m \leq \sum_{n,k} (1 + \epsilon_2)^{-1}\alpha_{n,k}^m u_{nk} + \epsilon_1 u.$$

Set $\lambda_{nk}^m = (\Sigma_{n,k}\alpha_{nk}^m)^{-1}\alpha_{nk}^m$ and $z_m = \Sigma_{n,k}\lambda_{nk}^m u_{nk}$; then, for every m, z_m is an element in $\langle\{u_{nk}\}\rangle$ and $(1 + \epsilon_2)^{-1}w_m \leq z_m + \epsilon_1 u$. Note that

$$(w_m - \epsilon u)^+ \leq [w_m - \epsilon(w_m + u)/2]^+ = [(1 + \epsilon_2)^{-1}w_m - \epsilon_1 u]^+;$$

hence $(1-\epsilon)v_m^\epsilon = (w_m - \epsilon u)^+ \leq z_m$ as required. This completes the proof of (2).

COROLLARY 3.4. *Let L be a Riesz space. Then $\rho_M = \rho_L$ for every monotone seminorm ρ on L if and only if L has the almost Egoroff property.*

Let ρ be a monotone seminorm on a Riesz space L. ρ is called strong Fatou if $\rho(u_\nu)\uparrow\rho(u)$ for every $u \in L^+$ and directed system $\{u_\nu\}$ in L satisfying $0 \leq u_\nu \uparrow u$. Every monotone seminorm ρ dominates a largest monotone seminorm ρ_m having the strong Fatou property, namely,

$$\rho_m(f) = \sup\{\rho'(f)\colon \rho' \text{ is a strong Fatou seminorm on } L \text{ such that } \rho' \leq \rho\}$$

for $f \in L$.

For every monotone seminorm ρ on L, we define a function ρ_l associated with ρ by:

$$\rho_l(f) = \inf\left\{\sup_\nu \rho(u_\nu)\colon\ 0 \leq u_\nu \uparrow |f|\right\}.$$

Then, ρ_l is again a monotone seminorm on L. Moreover, $\rho \geq \rho_l \geq \rho_m$, and $\rho = \rho_l$ if and only if ρ is strong Fatou. Since $\rho \geq \rho_m$, we have $\rho_l \geq \rho_{ml} = \rho_m$, so $\rho_m = \rho_l$ if and only if $\rho_l = \rho_{ll}$.

If a monotone seminorm ρ on L is of the form $\rho(f) = \phi(|f|)$ where ϕ is a positive linear functional on L, then $\rho_m = \rho_l$ (see [7, Theorem 57.4, Note XV_B). In general, it is not known how to construct ρ_m explicitly in terms of ρ. We shall next give a necessary and sufficient condition for a Riesz space L to have the property that $\rho_m = \rho_l$ for every monotone seminorm ρ on L.

DEFINITION 3.5. An element f of a Riesz space L is said to have the generalized almost Egoroff property if, whenever $0 \leqslant u_{\kappa_\nu} \uparrow u_\nu \uparrow |f|$ and $0 < \epsilon < 1$, there exists a directed system $\{v_\mu^\epsilon\}$ in L such that $0 \leqslant v_u^\epsilon \uparrow_\mu |f|$ and, for every μ, $(1-\epsilon)v_\mu^\epsilon \leqslant z_\mu$ for some element z_μ in $\langle\{u_{\kappa_\nu}\}\rangle$. A Riesz space L is said to have the generalized almost Egoroff property if every one of its elements has the generalized almost Egoroff property.

THEOREM 3.6. *Let L be a Riesz space and $u \in L^+$.*

(1) *If u has the generalized almost Egoroff property, then $\rho_m(u) = \rho_l(u)$ for every monotone seminorm ρ on L.*

(2) *If $\rho_m(u) = \rho_l(u)$ for every monotone seminorm ρ on L such that $\rho(u) < \infty$, then u has the generalized almost Egoroff property.*

PROOF. Exact analogue of the proof of Theorem 3.3; sequences are replaced by directed sets everywhere.

COROLLARY 3.7. *Let L be a Riesz space. Then $\rho_m = \rho_l$ for every monotone seminorm ρ on L if and only if L has the generalized almost Egoroff property.*

4. **σ-Fatou property of a monotone seminorm on a Riesz space.** In this section, we focus our attention on a given monotone seminorm ρ on an arbitrary Riesz space L and obtain necessary and sufficient conditions for ρ as well as ρ_L to be σ-Fatou, in terms of the order topology and the r.u. (relative uniform) topology of L. To be complete, we shall include the definitions of the order and r.u. topologies of a Riesz space. For further discussion of these topologies, the reader is referred to [9], [5], and [2].

A sequence $\{f_n\}$ in a Riesz space L is said to converge in order to an element $f \in L$ if there exists a sequence $\{u_n\}$ in L^+ such that $u_n \downarrow 0$ and $|f_n - f| \leqslant u_n$ for all n; this will be denoted by $f_n \longrightarrow f$. A subset S of L is called order closed if for every order convergent sequence in S the order limit of the sequence is also a member of S. The order topology of L is the topology which has as its closed sets the family of all order closed sets of L. For any subset S of L, the set of all $f \in L$ with the property that there exists a sequence in S converging in order to f is called the pseudo order closure of S and will be denoted by S'. The order closure, i.e., the closure in the order topology, of S will be denoted by $\overline{S}$. A sequence $\{f_n\}$ in L is said to converge r.u. (relatively uniformly) to an element $f \in L$ if there exists an element $u \in L^+$ and a real sequence $\epsilon_n \downarrow 0$ such that $|f_n - f| \leqslant \epsilon_n u$ for all n; this will be denoted by $f_n \xrightarrow{\text{ru}} f$. The r.u. closed set, r.u. topology, pseudo r.u. closure (denoted by S'_{ru}) and r.u. closure (denoted by $\overline{S}^{\text{ru}}$) are defined similarly.

A monotone seminorm ρ on a Riesz space L is called a Riesz seminorm if $\rho(f) < \infty$ for all $f \in L$. A linear subspace S of L is called an ideal of L whenever S is solid, i.e., whenever it follows from $f \in S, g \in L$ and $|g| \leqslant |f|$ that $g \in S$. If ρ is a monotone seminorm on L, then the set $L_\rho = \{f \in L: \rho(f) < \infty\}$ is an ideal of L and the restriction of ρ on L_ρ is a Riesz seminorm.

THEOREM 4.1. *Let* ρ *be a Riesz seminorm on a Riesz space* L. *Then* $f \xrightarrow{\text{ru}} f$ *implies that* $\rho(f_n - f) \longrightarrow 0$, *and so in particular* $\rho(f_n) \longrightarrow \rho(f)$.

PROOF. Let $f_n \xrightarrow{\text{ru}} f$; then there exists $u \in L^+$ and a real sequence $\epsilon_n \downarrow 0$ such that $|f_n - f| \leqslant \epsilon_n u$ for all n. Hence $\rho(f_n - f) \longrightarrow 0$. Since $\rho(f) - \rho(f_n - f) \leqslant \rho(f_n) \leqslant \rho(f) + \rho(f_n - f)$, so $\rho(f_n - f) \longrightarrow 0$ implies that $\rho(f_n) \longrightarrow \rho(f)$.

THEOREM 4.2. *A monotone seminorm* ρ *on a Riesz space* L *is* σ*-Fatou if and only if* ρ *is lower semicontinuous with respect to order convergence (i.e.* $f_n \longrightarrow f$ *implies* $\rho(f) \leqslant \liminf \rho(f_n)$).

PROOF. Assume that ρ is σ-Fatou. Let $f_n \longrightarrow f$. Then there exists $u_n \downarrow 0$ such that $|f_n - f| \leqslant u_n$ for all n, and so the sequence $w_n = (|f| - u_n)^+$ satisfies $0 \leqslant w_n \uparrow |f|$ and $w_n \leqslant |f_n|$ for all n. Hence, by the σ-Fatou property of ρ, $\rho(f) \leqslant \liminf \rho(f_n)$. Conversely, assume that $\rho(f) \leqslant \liminf \rho(f_n)$ whenever $f_n \longrightarrow f$. Let $0 \leqslant u_n \uparrow u$. Then $\rho(u) \leqslant \liminf \rho(u_n) = \lim \rho(u_n)$ and so $\rho(u_n) \uparrow \rho(u)$. Hence, ρ is σ-Fatou.

For every monotone seminorm ρ on a Riesz space L, the set of all $f \in L$ such that $\rho(f) \leqslant 1$ will be denoted by S_ρ. It follows that S_ρ is convex and solid; moreover, ρ is a Riesz seminorm if and only if S_ρ is absorbent (i.e. for each $f \in L$ there exists some real $\alpha > 0$ such that $f \in \beta S_\rho$ for all $\beta \geqslant \alpha$).

THEOREM 4.3. *A monotone seminorm* ρ *on a Riesz space* L *is* σ*-Fatou if and only if* S_ρ *is order closed.*

PROOF. Assume that ρ is σ-Fatou. Let $f_n \longrightarrow f$ and $f_n \in S_\rho$ for all n. Then, by Theorem 4.2, $\rho(f) \leqslant \liminf \rho(f_n) \leqslant 1$. Hence $f \in S_\rho$. Conversely, assume that S_ρ is order closed. Let $0 \leqslant u_n \uparrow u$. Clearly, $\rho(u_n)\uparrow \leqslant \rho(u)$. Let α be such that $\rho(u_n) < \alpha$ for all n. Then $\alpha^{-1} u_n \in S_\rho$ for all n; and so, by the assumption that S_ρ is order closed, $\alpha^{-1} u \in S_\rho$, i.e., $\rho(u) \leqslant \alpha$. It follows that $\rho(u_n) \uparrow \rho(u)$ and hence ρ is σ-Fatou.

For every convex, solid subset S of a Riesz space, the Minkowski functional of S will be denoted by ψ_S, i.e., $\psi_S(f) = \inf\{\alpha > 0: f \in \alpha S\}$.

THEOREM 4.4. *If* ρ *is a monotone seminorm on a Riesz space* L, *then* ρ_L *is the Minkowski functional of the pseudo order closure of* S_ρ.

PROOF. Let $u \in L^+$. If α is such that $\rho_L(u) < a < \infty$, then there exists $0 \leqslant u_n \uparrow u$ such that $\rho(u_n) < \alpha$ for all n; and so $\alpha^{-1}u_n \in S_\rho$ for all n, thus $\alpha^{-1}u \in S'_\rho$ and $\psi_{S'_\rho}(u) \leqslant \alpha$. This shows that $\psi_{S'_\rho} \leqslant \rho_L$. Conversely, if β is such that $\psi_{S'_\rho}(u) < \beta < \infty$, then $u \in \beta S'_\rho$; hence there exists $0 \leqslant u_n \uparrow u$ such that $u_n \in \beta S_\rho$ for all n, so $\rho_L(u) \leqslant \beta$. Therefore $\rho_L \leqslant \psi_{S'_\rho}$.

THEOREM 4.5. *Let* ρ *be a monotone seminorm on a Riesz space* L. *Then the largest* σ*-Fatou monotone seminorm* ρ_M *dominated by* ρ *is the Minkowski functional of the order closure of* S_ρ.

PROOF. Denote by ψ the Minkowski functional of the order closure of S_ρ. If $0 \leqslant u_n \uparrow u$ and $\psi(u_n) < \alpha < \infty$ for all n, then $\alpha^{-1}u_n \in \overline{S}_\rho$ for all n, so $\alpha^{-1}u \in \overline{S}$ and $\psi(u) \leqslant \alpha$. This shows that ψ is σ-Fatou. If η is a σ-Fatou monotone seminorm such that $\eta \leqslant \rho$, then $S_\rho \subseteq S_\eta$ and so $\overline{S}_\rho \subseteq \overline{S}_\eta = S_\eta$. Let α be such that $\psi(f) < \alpha < \infty$, then $\alpha^{-1}f \in \overline{S}_\rho \subseteq S_\eta$, hence $\eta(f) \leqslant \alpha$. Thus $\eta \leqslant \psi$. Therefore $\psi \equiv \rho_M$.

By Theorems 4.4 and 4.5, S'_ρ is order closed implies that ρ_L is σ-Fatou. The following theorem gives a necessary and sufficient condition for ρ_L to be σ-Fatou.

THEOREM 4.6. *Let* ρ *be a monotone seminorm on a Riesz space* L. *Then* ρ_L *is* σ*-Fatou if and only if the* ρ_L*-closure of* S'_ρ *is order closed.*

PROOF. By Theorem 4.3, ρ_L is σ-Fatou if and only if S_{ρ_L} is order closed. Hence, it is sufficient to show that the ρ_L-closure of S'_ρ is S_{ρ_L}. Denote the ρ_L-closure of S'_ρ by $\overline{S'_\rho}^{\rho_L}$. Since S_{ρ_L} is ρ_L-closed and $S'_\rho \subseteq S_{\rho_L}$, so $\overline{S'_\rho}^{\rho_L} \subseteq S_{\rho_L}$. For the reverse inclusion, let $f \in S_{\rho_L}$. Then $\rho_L(f) \leqslant 1$, and so $g_n = (1 - 1/n)f \in S'_\rho$ for all $n = 1, 2, \cdots$ and $\rho_L(g_n - f) \longrightarrow 0$. Thus $f \in \overline{S'_\rho}^{\rho_L}$.

If ρ is a Riesz seminorm, then the property that ρ_L is σ-Fatou can be interpreted in a different way than Theorem 4.6. Before proving this, we shall first prove the following two lemmas.

LEMMA 4.7. *Let* S *be a convex solid subset of a Riesz space* L. *Then* $(\psi_S)_L = \psi_{S'}$.

PROOF. Similar to the proof of Theorem 4.4.

LEMMA 4.8. *Let* S, T *be two convex solid absorbent subsets of a Riesz space* L. *Then* $\psi_S = \psi_T$ *if and only if* $S'_{\mathrm{ru}} = T'_{\mathrm{ru}}$.

PROOF. Assume that $\psi_S = \psi_T$. Let u be an element such that $0 \leqslant$

$u \in S'_{\mathrm{ru}}$. There exists then a sequence $\{u_n\}$ such that $u_n \xrightarrow{\mathrm{ru}} u$ and $0 \leq u_n \in S$ for all n. Since ψ_S is a Riesz seminorm, it follows from Theorem 4.1 that $\psi_S(u_n) \to \psi_S(u)$ and so $\psi_S(u) \leq 1$. Then, by assumption, $\psi_T(u) \leq 1$ and hence $u \in T'_{\mathrm{ru}}$. This shows that $S'_{\mathrm{ru}} \subseteq T'_{\mathrm{ru}}$. Similarly, $T'_{\mathrm{ru}} \subseteq S'_{\mathrm{ru}}$. Therefore, $S'_{\mathrm{ru}} = T'_{\mathrm{ru}}$. For the reverse implication, assume that $S'_{\mathrm{ru}} = T'_{\mathrm{ru}}$. Let $u \in L^+$. If $\psi_T(u) < \alpha < \infty$, then $\alpha^{-1}u \in T$ and so $\alpha^{-1}u \in S'_{\mathrm{ru}}$. There exists a sequence $\{u_n\}$ such that $u_n \xrightarrow{\mathrm{ru}} \alpha^{-1}u$ and $0 \leq u_n \in S$ for all n. Then, by Theorem 4.1, $\psi_S(u_n) \to \psi_S(\alpha^{-1}u)$ and hence $\psi_S(u) \leq \alpha$. This shows that $\psi_S \leq \psi_T$. Similarly, $\psi_T \leq \psi_S$. Therefore, $\psi_S = \psi_T$.

THEOREM 4.9. *Let ρ be a Riesz seminorm on a Riesz space L. Then ρ_L is σ-Fatou if and only if $(S'_\rho)'_{\mathrm{ru}} = (S''_\rho)'_{\mathrm{ru}}$.*

PROOF. From Theorem 4.4, $\rho_L = \psi_{S'_\rho}$. Then $\rho_{LL} = (\psi_{S'_\rho})_L$ and, by Lemma 4.7, $\rho_{LL} = \psi_{S''_\rho}$. Hence, Lemma 4.8 implies that ρ_L is σ-Fatou if and only if $(S'_\rho)'_{\mathrm{ru}} = (S''_\rho)'_{\mathrm{ru}}$.

For every subset S of a Riesz space L, we define $S^n, n = 1, 2, \cdots$, inductively by: $S = S', S^n = (S^{n-1})'$; for every monotone seminorm ρ on L, define ρ_{Ln} by: $\rho_{L1} = \rho_L, \rho_{Ln} = (\rho_{Ln-1})_L$. The following theorem is a generalization of Theorem 4.9.

THEOREM 4.10. *For each $n = 1, 2, \cdots$,*

(1) *if ρ is a Riesz seminorm on a Riesz space L, then ρ_{Ln} is σ-Fatou if and only if $(S^n_\rho)'_{\mathrm{ru}} = (S^{n+1}_\rho)'_{\mathrm{ru}}$;*

(2) *if ρ is a Riesz seminorm on an Archimedean Riesz space L, then ρ_{Ln} is σ-Fatou implies that S^{n+1}_ρ is order closed.*

PROOF. (1) Since $\rho_L = \psi_{S'_\rho}$ and $(\psi_T)_L = \psi_T$, for every convex solid subset T, so by induction $\rho_{Ln} = \psi_{S^n_\rho}$ for $n = 1, 2, \cdots$. From Lemma 4.8, it follows that ρ_{Ln} is σ-Fatou if and only if $(S^n_\rho)'_{\mathrm{ru}} = (S^{n+1}_\rho)'_{\mathrm{ru}}$.

(2) If L is Archimedean, then

$$(S^n_\rho)'_{\mathrm{ru}} \subseteq S^{n+1}_\rho \subseteq (S^{n+1}_\rho)'_{\mathrm{ru}} \subseteq S^{n+2}_\rho \subseteq (S^{n+2}_\rho)'_{\mathrm{ru}}.$$

By part (1) and the σ-Fatou property of ρ_{Ln} and ρ_{Ln+1},

$$(S^n_\rho)'_{\mathrm{ru}} = (S^{n+1}_\rho)'_{\mathrm{ru}}, \qquad (S^{n+1}_\rho)'_{\mathrm{ru}} = (S^{n+2}_\rho)'_{\mathrm{ru}}.$$

Hence $S^{n+1}_\rho = S^{n+2}_\rho$ and so S^{n+1}_ρ is order closed.

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